# Equivariant Kazhdan–Lusztig theory of paving matroids

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(joint with George Nasr, Nick Proudfoot, and Lorenzo Vecchi)
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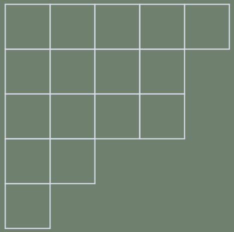
# » Overview

# A classical story

A <u>partition</u> $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to $n$.

**Example**

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A Young tableau $T$ is a filling of a Ferrers diagram by positive integers. $T$ is standard if it is filled by $\{1, 2, \ldots, n\}$ and increasing in rows and columns. Define $f_\lambda$ as the number of standard tableaux of shape $\lambda$.

**Example**

One of $f_{(5,4,4,2,1)} = 549120$ standard Young tableaux:

| | | | | |
|---|---|---|---|---|
| 1 | 6 | 10 | 13 | 16 |
| 2 | 7 | 11 | 14 | |
| 3 | 8 | 12 | 15 | |
| 4 | 9 | | | |
| 5 | | | | |

**Fact**

Fix $n$. Then
$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Proof 1:
The Robinson-Schensted bijection:

pairs of standard tableaux of same shape $\longleftrightarrow$ symmetric group $\mathfrak{S}_n$

**Fact**

Irreducible $\mathfrak{S}_n$ representations are indexed by $\lambda \vdash n$ and have dimension $f_\lambda$.

**Fact**

Let $d_1, d_2, \ldots, d_r$ be the dimensions of the irreducible complex representations of a finite group. Then

$$\sum_i d_i^2 = |G|.$$

**Fact**

Fix $n$. Then

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Proof 2:

$$\sum_\lambda f_\lambda^2 = \sum_i d_i^2 = |G| = |\mathfrak{S}_n| = n!$$

Combinatorial identity with standard Young tableaux

Combinatorial proof

Algebraic proof

Show me the matroids!

**Definition 1**

A **matroid** $M = (E, \mathcal{I})$ is a set $E$ with $\mathcal{I} \subseteq 2^E$ such that

- If $I \in \mathcal{I}$ then all subsets of $I$ are in $\mathcal{I}$, and
- If $I_1, I_2 \in \mathcal{I}$ and $|I_1| = |I_2| + 1$, there exists $x \in I_1 - I_2$ such that $I_2 \cup x \in \mathcal{I}$

Elements of $\mathcal{I}$ are **independent sets**. The **bases** of $M$ are the inclusion-maximal elements of $\mathcal{I}$. The set of all bases is $\mathcal{B}$.

## Definition 2

A **matroid** $M = (E, \mathcal{C})$ is a set $E$ with $\mathcal{C} \subseteq 2^E$ such that

- $\emptyset \notin \mathcal{C}$
- If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C_1 = C_2$.
- If $C_1, C_2 \in \mathcal{C}$ are distinct, and $e \in C_1 \cap C_2$, then there is a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

Elements of $\mathcal{C}$ are **circuits**. A circuit of $M$ is an minimal set which is not in $\mathcal{I}$.

### Example

The **uniform matroid** $U_{k,n}$ models $n$-many vectors in $\mathbb{R}^k$ in general position

Bases $\longleftrightarrow$ any set of $k$-many vectors

Circuits $\longleftrightarrow$ any set of $k+1$-many vectors

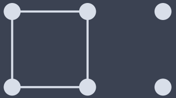
## Example

A graph $\Gamma$ with edges $E$ forms a matroid:

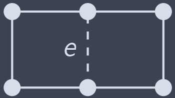
Bases $\longleftrightarrow$ spanning trees

Circuits $\longleftrightarrow$ cycles

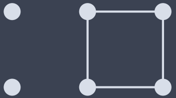
$C_1$



$C_3$




$C_2$

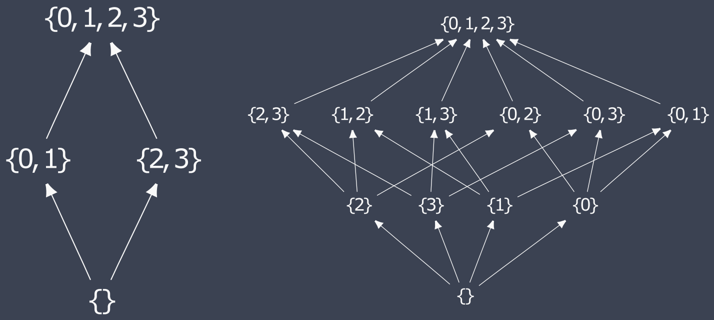


## Example

The columns of a matrix form a matroid:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

The combinatorial model for a subspace is a **flat**




The flats of a matroid form a ranked lattice. The rank of the matroid is then defined to be the rank of the lattice. A rank-$k-1$ flat is called a **hyperplane.** If a hyperplane $H \in \mathcal{C}$, then it is called a **circuit hyperplane.**

Let $x, y$ be elements of a poset. Define the Möbius function

$$\mu(x,y) := \begin{cases} 1 & x = y \\ -\sum_{x \le z < y} \mu(x,z) & \text{otherwise} \end{cases}$$

Let $M$ be a rank-$k$ matroid with lattice of flats $L(M)$

$$\chi_M(t) := \sum_{F \in L(M)} \mu(\overline{\emptyset}, F) t^{k-r(F)}$$

A matroid $M$ on groundset $E$, has Orlik–Solomon algebra $\mathcal{OS}(M)$, a certain quotient of the exterior algebra $\bigwedge E$

**Theorem (Orlik, Solomon '80)**

$\chi_M(t)$ determines the Poincaré polynomial of $\mathcal{OS}(M)$

**Definition/Theorem (Elias, Proudfoot, Wakefield '16)**

Fix $M$. There is a unique polynomial $P_M(t)$ satisfying:

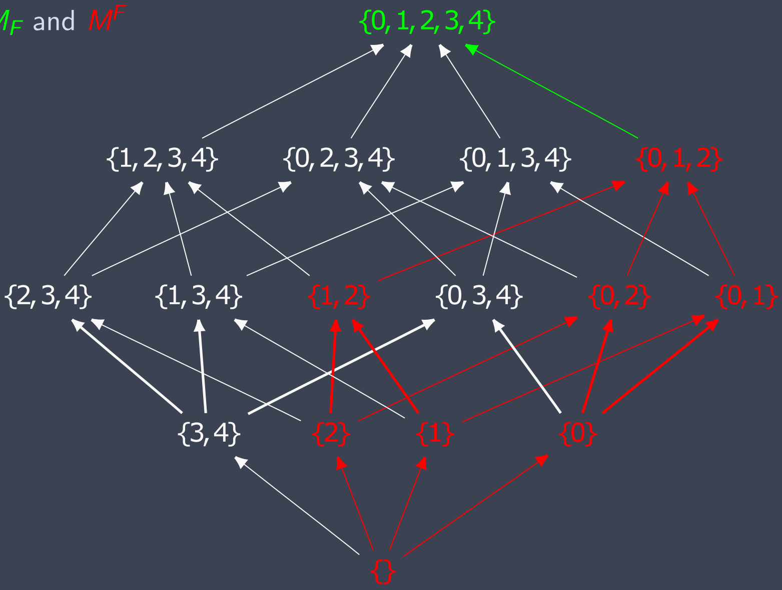
$$P_M(t) = 1 \text{ if } r(M) = 0,$$

$$\deg P_M(t) < r(M)/2 \text{ when } r(M) > 0,$$

$$t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} P_{M_F}(t) \chi_{M^F}(t).$$

$P_M(t)$ is the matroid Kazhdan–Lusztig polynomial

$M_F$ and $M^F$



**Conjecture (Elias, Proudfoot, Wakefield '16)**

$P_M(t)$ has positive coefficients

**Theorem (Lee, Nasr, Radcliffe '21)**

Conjecture true for sparse paving matroids

**Theorem (Braden, Huh, Matherne, Proudfoot, Wang '20)**

Conjecture true for any $M$

$M$ is a paving matroid if all circuits have size at least $k = r(M)$

A paving matroid is sparse if the set $\mathcal{CH}$ of circuit hyperplanes satisfies $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

**Conjecture (Mayhew, Newman, Welsh, Whittle '11)**

Asymptotically almost all matroids are sparse paving

**Theorem (Pendavingh, van der Pol '15)**

Asymptotically logarithmically almost all matroids are sparse paving

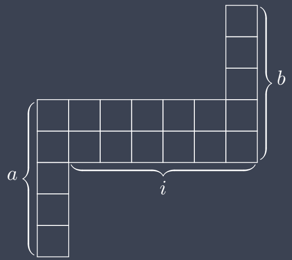
**Theorem (Lee, Nasr, Radcliffe '21)**

Let $M$ be rank-$k$, sparse paving, on a groundset of size $n$, with circuit hyperplanes $\mathcal{CH}$. The $t^i$ coefficient in $P_M(t)$ is

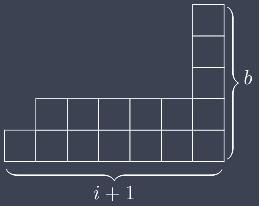
$$\mathrm{SSkYT}(n-k+1, i, k-2i+1) - |\mathcal{CH}| \cdot \overline{\mathrm{SSkYT}}(i, k-2i+1)$$

Proof idea: Combinatorial argument with recursion.

$\mathsf{SSkYT}(a, i, b) = \#$standard fillings of




$\overline{\mathsf{SSkYT}}(i, b) = \#$standard fillings of



Lee, Nasr, and Radcliffe combinatorially prove an identity involving Kazhdan-Lusztig polynomials and standard fillings of skew Young tableaux.

**Fact**

Standard skew Young tableaux count the dimension of certain (reducible) $\mathfrak{S}_n$ representations

**Question**

Is there representation theory lurking?

**K., Nasr, Proudfoot, Vecchi '22**

YES!

# Main result

Let $W$ be a group. An equivariant matroid $W \curvearrowright M$ is a matroid with a $W$-action such that $gI \in \mathcal{I}$ for all $g \in W$ and $I \in \mathcal{I}$.

The action of $W$ induces an action on $\mathcal{OS}(M)$. The equivariant characteristic polynomial of $W \curvearrowright M$ is a graded virtual representation $\chi_M^W(t)$. The coefficient of $t^{k-i}$ is determined by $\mathcal{OS}(M)_i$.

$$\boxed{\chi_M^W(t)} \xrightarrow{\text{“dimension”}} \boxed{\chi_M(t)}$$

## Definition/Theorem (Gedeon, Proudfoot, Young '17)

Let $W \curvearrowright M$ be an equivariant matroid, $W_F$ denote the stabilizer of $F$. Then there exists $P_M^W(t)$ satisfying

If $r(M) = 0$, then $P_M^W(t)$ is $\mathbb{1}_W t^0$

If $r(M) > 0$, then $\deg P_M^W(t) < r(M)/2$

$$t^{r(M)} P_M^W(t^{-1}) = \sum_{[F] \in L(M)/W} \mathrm{Ind}_{W_F}^W \left( P_{M_F}^{W_F}(t) \otimes \chi_{M^F}^{W_F} \right)$$

$\varphi : W' \to W$ a homom. then $P_M^{W'}(t) = \varphi^* P_M^W(t)$

Compare:

$$t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} P_{M_F}(t) \chi_{M^F}(t)$$

and

$$t^{r(M)} P_M^W(t^{-1}) = \sum_{[F] \in L(M)/W} \mathrm{Ind}_{W_F}^{W} \left( P_{M_F}^{W_F}(t) \otimes \chi_{M^F}^{W_F} \right).$$

$$\boxed{P_M^W(t)} \xrightarrow{\text{dimension}} \boxed{P_M(t)}$$

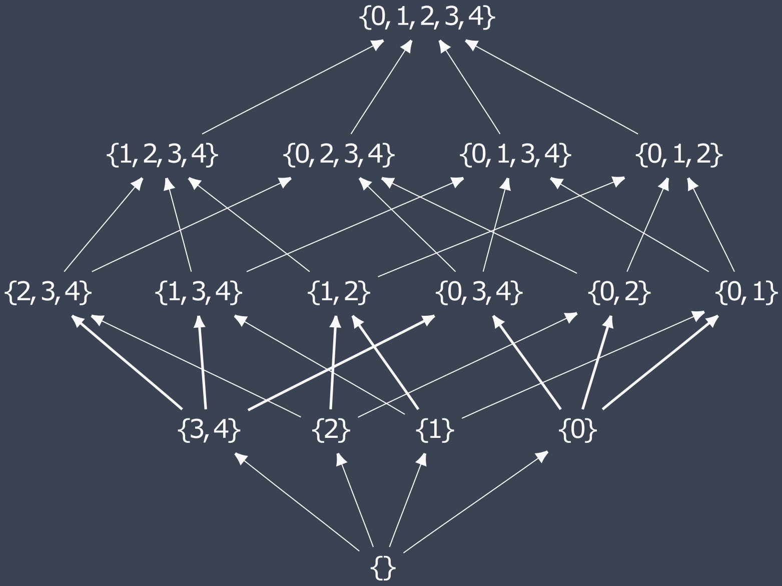
# » Relaxation

A stressed hyperplane $H$ of a rank-$k$ matroid $M = (E, B)$ has every $k$-subset a circuit.

**Theorem (Ferroni, Nasr, Vecchi '21)**

The operation of relaxation at a stressed hyperplane $H$ forms a new matroid $\tilde{M} = (E, \tilde{\mathcal{B}})$ with bases

$$\tilde{\mathcal{B}} = \mathcal{B} \sqcup \{S \subseteq H : |S| = k\}.$$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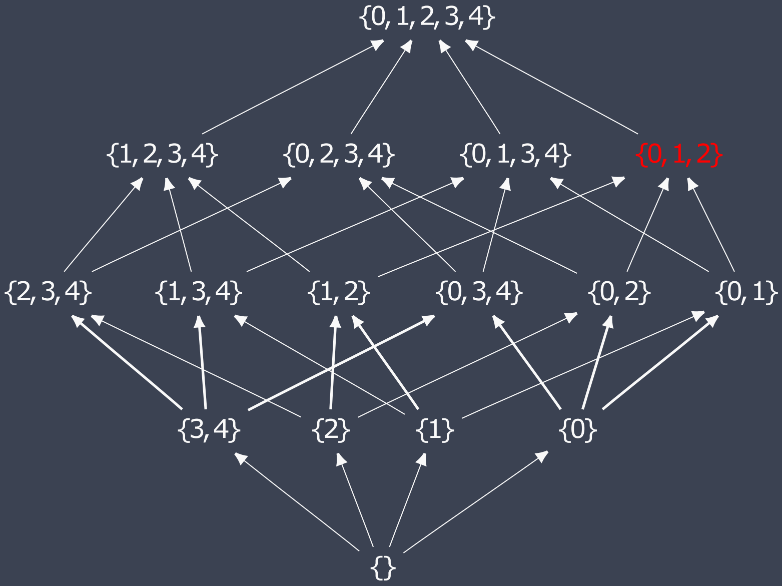
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**Theorem (Ferroni, Nasr, Vecchi '21)**

There exists a polynomial $p_{k,h}$ such that

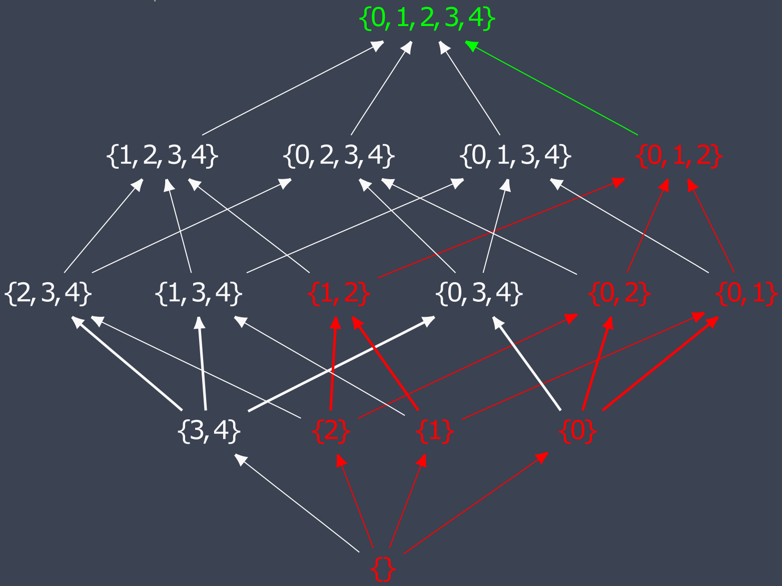
$$P_M(t) = P_{\tilde{M}}(t) - p_{k,h}$$

**Theorem (Ferroni, Nasr, Vecchi '21)**

If $M$ is a paving matroid with $|E| = n$ and has exactly $\lambda_h$-many stressed hyperplanes of size $h$, then

$$P_M(t) = P_{U_{k,n}}(t) - \sum_{h \geq k} \lambda_h \cdot p_{k,h}.$$

# » Idea of the proof



Let $W \curvearrowright M$ be an equivariant matroid with stressed hyperplane $H$.

Let $W \curvearrowright \widetilde{M}$ denote the equivariant matroid found by simultaneously relaxing all hyperplanes in $[H]$.

**Theorem (K.-Nasr-Proudfoot-Vecchi '22)**

There exists an equivariant polynomial $p_{k,h}^{\mathfrak{S}_h}$ such that

$$P_M^W(t) = P_{\widetilde{M}}^W(t) - \mathrm{Ind}_{W_H}^W \mathrm{Res}_{W_H}^{\mathfrak{S}_h} p_{k,h}^{\mathfrak{S}_h}$$

**Theorem (K.-Nasr-Proudfoot-Vecchi '22)**

The coefficients of $t^i$ are

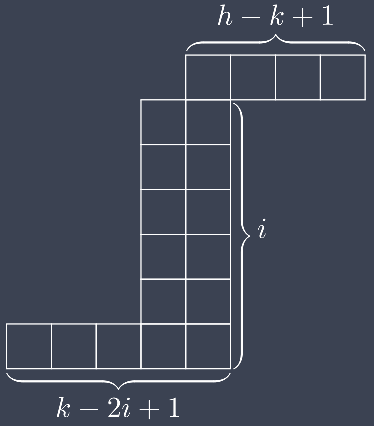
$$\{t^i\}p_{k,h}^{\mathfrak{S}_h} = S^{\mu_i/\lambda_i}$$

where $\mu_i, \lambda_i \vdash h$ are:

$\mu_i = h - 2i + 1, (k - 2i + 1)^i$ and

$\lambda_i = k - 2i, (k - 2i - 1)^{i-1}$

The coefficient of $t^i$ in $p_{k,h}^{\mathfrak{S}_h}$ is




which has dimension equal to the number of standard fillings

## » Idea of proof

Relax $U_{k-1,h}^{\mathfrak{S}_h} \oplus U_{1,1}$ to $U_{k,h+1}^{\mathfrak{S}_{h+1}}$.

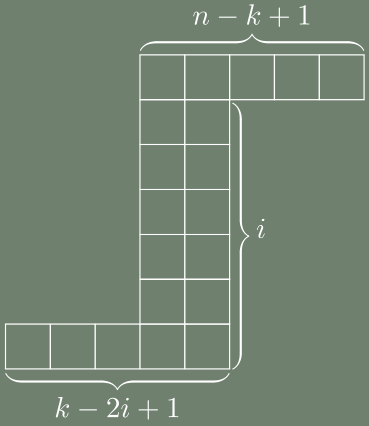
$p_{k,h}^{\mathfrak{S}_h}$ depends only on $k, h$, so one example is enough.

A series of relaxations can be performed to a sparse paving matroid to obtain the uniform matroid. In other words:

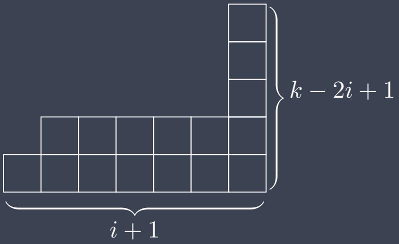
$$P_M^W(t) = P_{U_{k,n}}^W(t) - \sum_{[H]\in\mathcal{CH}} \operatorname{Ind}_{W_H}^W \operatorname{Res}_{W_H}^{\mathfrak{S}_h} p_{k,h}^{\mathfrak{S}_h}$$

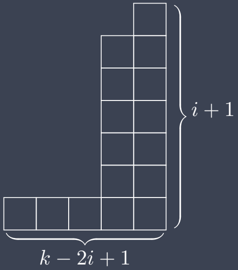
## Theorem (Gao, Xie, Yang '21)

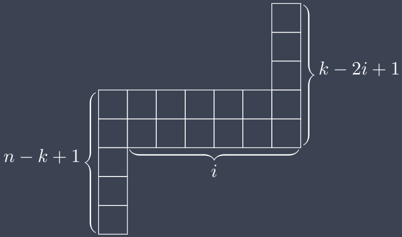
Every coefficient of $t^i$ in $P^{\mathfrak{S}_n}_{U_{k,n}}(t)$ is given by the skew shape:

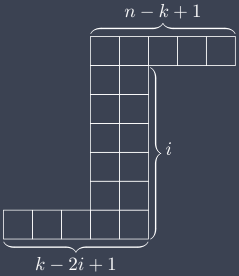


For sparse paving matroids, $h = k$. This provides representation theoretic proof of the Lee–Nasr–Radcliffe formula!



$$\sum_{\lambda} f_{\lambda}^{2} = n!$$

Combinatorial proof

Algebraic proof

$P_M(t)$ in terms of standard skew Young tableaux

Combinatorial proof [LNR21]

Algebraic proof [KNPV22]

# THANK YOU!

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