Applying hyperplane arrangements to study superspace

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» Overview

Finding linear bases

Superspace and Sagan-Swanson

Free arrangements and Solomon-Terao algebras

Finding linear bases

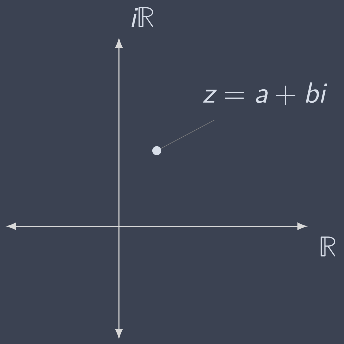
Main problem: find a linear basis for a quotient ring.

A basis for a vector space is a set of

- linearly independent vectors,
- that span

» Example

$\mathbb{C} \cong \mathbb{R}^2$ as a vector space with basis $\{1, i\}$.



Wrinkle: $\mathbb{C}$ has a multiplication that is not coordinatewise.
Fix: construct $\mathbb{C}$ as a quotient ring.

Informal Definition

A ring is an abstract number system where you are allowed to add, subtract, and multiply, but not necessarily divide.

Example

The integers are a ring.

Example

All polynomials with real-number coefficients form a ring called $\mathbb{R}[x]$.

$$(2018x + 2025) + (7x - 7) = 2025x + 2018$$

$$(2018x+2025)(x^2+1) = 2018x^3+2025x^2+2018x+2025$$

One special number in a ring is 0.

0 is like a black hole for multiplication.

But it is not the only black hole:

Definition

A subset $I \subseteq R$ is an ideal if for all $y \in R$,

$$x \in I \Rightarrow xy \in I.$$

Example

Let $I = \langle x^2 + 1 \rangle$.

We have already seen an element of I:
$2018x^3 + 2025x^2 + 2018x + 2025 = (2018x + 2025)(x^2 + 1)$.

Informal definition

A quotient ring R/I is a ring obtained by setting every element of $I = 0$.

Example

Let $I = \langle x^2 + 1 \rangle$. In $\mathbb{R}[x]/I$,

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \sqrt{-1} = i,$$

so $\mathbb{C} = \mathbb{R}[x]/I$.

$\mathbb{R}[x]$ is a vector space, with basis $\{1, x, x^2, x^3, x^4, \ldots\}$.

A basis for $\mathbb{C}$ from a basis for $\mathbb{R}[x]$?

$$\begin{aligned} 1 &\to 1 \\ x &\to i \\ x^2 &\to -1 \\ x^3 &\to -i \\ x^4 &\to 1 \\ x^5 &\to i \\ &\vdots \end{aligned}$$

Only get $\{1, i, -1, -i\}$. Remove dependencies to get $\{1, i\}$.

A general mathematical game we can play: take a quotient ring that also has the structure of a vector space, and find a basis

Consider polynomials in several variables:

$$S = \mathbb{R}[x_1, x_2, \ldots, x_n]$$

Some polynomials are "symmetric":

$$x_1x_2 + x_1x_3 + x_2x_3 \qquad \longleftrightarrow \qquad x_2x_1 + x_2x_3 + x_1x_3$$

Definition

Let $S_+^{\mathfrak{S}_n}$ denote the ideal of $\mathbb{R}[x_1, \ldots, x_n]$ generated by symmetric polynomials with constant term 0.

Example

$$x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2 \in S_+^{\mathfrak{S}_n}$$

but not symmetric.

Definition

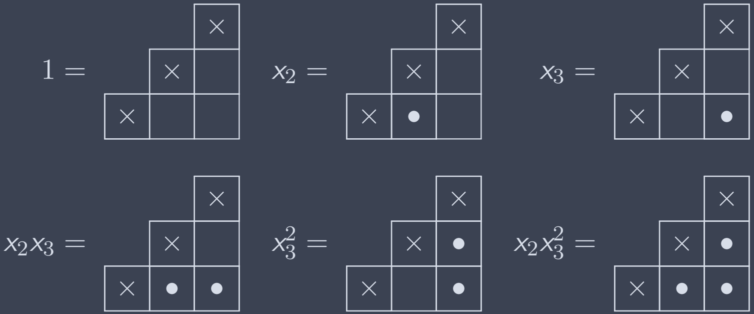
$R_n = \mathbb{R}[x_1, \ldots, x_n]/S_+^{\mathfrak{S}_n}$ is called the coinvariant ring.

Two famous bases for R_n:

- the "Artin staircase basis" and
- the "Schubert polynomials".

» Example

The Artin basis for R_3 is given by "substaircase monomials":



Proof idea: similar to $\mathbb{C}$. Rewrite $x_1 = -x_2 - x_3$ so we can rewrite x_1 in terms of others. Similarly, rewrite x_2^2, etc.

Superspace and Sagan-Swanson

Main problem: find a basis for the superspace coinvariant ring

Definition

Superspace is

$$\Omega = \mathrm{Sym}(\mathbb{K}^N) \otimes_{\mathbb{K}} \wedge \mathbb{K}^n.$$

Polynomials with variables: $x_1, \ldots, x_n$ and $\theta_1, \ldots, \theta_n$.

- x_i variables commute with everything.
- θ_j variables "anti-commute":

$$\theta_i \theta_j = -\theta_j \theta_i$$

Example

$$(x_1\theta_2 + x_2\theta_1 + x_3\theta_2\theta_3)\, x_1\theta_1 = -x_1^2\theta_1\theta_2 + 0 + x_1x_3\theta_1\theta_2\theta_3$$

Ω can have polynomials that are symmetric too.

Examples

$$x_1 + x_2 + x_3$$

$$\theta_1 + \theta_2 + \theta_3$$

$$x_1\theta_1 + x_2\theta_2 + x_3\theta_3$$

$$x_1x_2x_3$$

Nonexample

$$\theta_1\theta_2\theta_3 = -\theta_2\theta_1\theta_3$$

Definition

Let $\Omega_+^{\mathfrak{S}_n}$ denote the ideal generated by invariant polynomials with 0 constant term. Then

$$SR_n = \Omega/\Omega_+^{\mathfrak{S}_n}$$

is the superspace coinvariant ring.

Example

Every Artin substaircase monomial is nonzero in SR_n. Also

$$\theta_1\theta_2\theta_3 = \theta_1(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) = 0 \in SR_n$$

Symmetric functions $\longleftrightarrow$ representation theory of $\mathfrak{S}_n$.

A common game: combinatorially describe a symmetric function

Example

The Schur functions s_λ are the generating functions for semistandard Young tableaux of shape λ.

The Delta Conjecture, roughly

There is a nice description of the symmetric function $\Delta'_{e_{n-1}} e_n$ in terms of labeled Dyck paths.

Final boss: representation theoretic justification

Conjecture [Zab19]: generalization of SR_n justifies Delta Conjecture.

Three variables is harder than two: Sagan and Swanson study SR_n.

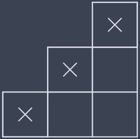
[SS24] want combinatorial descriptions of the dimensions of SR_n.

Definition

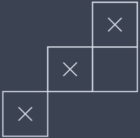
Let $J \subseteq [n]$. A J-penalized staircase is a diagram where the columns indexed by $j \in J$ are the same height as the column to the left, and if $k \notin J$ then the kth column is one larger than the previous.

» Example

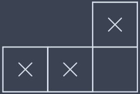
$\emptyset =$



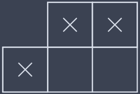
$\{1\} =$



$\{2\} =$



$\{3\} =$



$\{2,3\} =$

Conjecture [SS24]/Theorem [ACK+24]

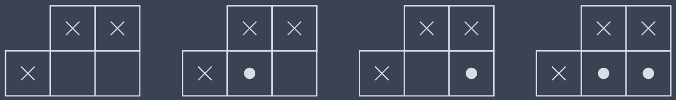
$$\mathcal{B} = \bigcup_{J \subseteq [n]} \{m\theta_J : m \text{ is a sub-}J\text{-penalized-staircase}\}$$

is a basis for SR_n.

Example

$J = \{3\}$

θ_3 $\quad$ $x_2\theta_3$ $\quad$ $x_3\theta_3$ $\quad$ $x_2x_3\theta_3$



Theorem [RW23]

The dimension of SR_n is the same as the size of $\mathcal{B}$.

What remains? Linear independence!

Free arrangements and Solomon-Terao algebras

Main problem: Show $\mathcal{B}$ is linearly independent, thus SR_n basis.

Definition

A hyperplane arrangement $\mathcal{A}$ is a union of $n-1$-dimensional linear subspaces H_i of $\mathbb{K}^n$.

Definition

The module $\mathrm{Der}(\mathcal{A})$ is the module of polynomial functions $p : \mathbb{K}^n \to \mathbb{K}^n$ such that $p(H_i) \subseteq H_i$ for all i.

Definition

An arrangement $\mathcal{A}$ is called a free arrangement if $\mathrm{Der}(\mathcal{A})$ is a free $\mathbb{K}[\mathrm{x}]$ module.

Definition

Let $\tilde{\mathcal{A}}_n$ denote the hyperplane arrangement defined as the zero set of the polynomial

$$x_1 x_2 \cdots x_n \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

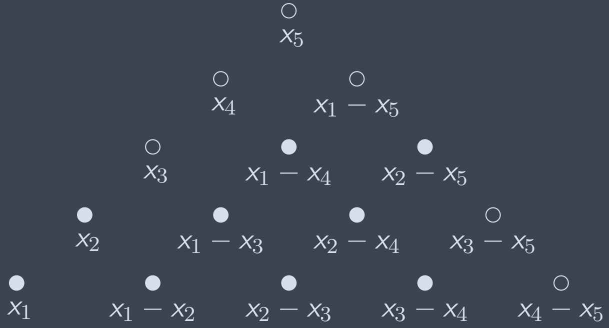
consisting of hyperplanes $H_{0,i} = x_i$ and $H_{i,j} = x_i - x_j$.

Definition

A subarrangement $\mathcal{A} \subseteq \tilde{\mathcal{A}}_n$ is a southwest arrangement if

$$H_{i,j} \in \mathcal{A} \text{ and } j > i + 1 \text{ imply } H_{i,j-1} \in \mathcal{A}.$$

» Example


x_5
x_4
$x_1 - x_5$
x_3
$x_1 - x_4$
$x_2 - x_5$
x_2
$x_1 - x_3$
$x_2 - x_4$
$x_3 - x_5$
x_1
$x_1 - x_2$
$x_2 - x_3$
$x_3 - x_4$
$x_4 - x_5$

Theorem [ACK+24]

Southwest arrangements are free.

Definition

Let $\varphi : \mathrm{Der}(\mathcal{A}) \to S$ be a linear map. Then define the Solomon-Terao algebra

$$\mathcal{ST}_\phi(\mathcal{A}) = S / \operatorname{im} \varphi$$

Theorem [AMMN19]

If $\mathcal{A}$ is a free arrangement (+ generic condition on ϕ), then $\mathcal{ST}_\phi(\mathcal{A})$ is a complete intersection.

Theorem [RW23] - "Transfer theorem"

There is a family of commutative quotient rings S/I_J such that if there is a set of monomials M_J which form a basis for S/I_J, then a basis for SR_n is

$$\bigcup_{J \subseteq [n]} \{m\theta_J : m \in M_J\}$$

Definition

Let $\mathcal{A}_J$ denote the subarrangement of $\tilde{\mathcal{A}}_n$ defined by

$$\mathcal{A}_J = \{H_{0,i} \ : \ i \notin J\} \cup \{H_{i,k} \ : i \notin J, \ k > i\}$$

NOT southwest in general

Theorem [ACK+24]

Let $1 : \mathrm{Der}(\mathcal{A}_J) \to S$ be defined by sending $\partial_i \mapsto 1$. Then

$$\mathcal{ST}_1(\mathcal{A}_J) = S/(S_+^{\mathfrak{S}_n} : f_J) = S/I_J$$

where

$$f_J = \prod_{j \in J} x_j \prod_{i > j} x_j - x_i$$

Basis for $\mathrm{Der}(\mathcal{A})$ for southwest $\mathcal{A} \subseteq \mathcal{A}_J$.

$\Downarrow$

Monomial basis for $\mathcal{ST}_1(\mathcal{A})$

$\Downarrow$

Injection $S/I_J \to \mathcal{ST}_1(\mathcal{A})$

$\Downarrow$

J-staircase M_J is linearly independent

$\Downarrow$

$\mathcal{B}$ is a basis

» Future directions

- Where do other $\mathcal{ST}_\phi(\mathcal{A})$ show up?
- Is there a space X such that $H^\bullet(X) \cong \mathcal{ST}_\phi(\mathcal{A})$?
- "What about type-B"?
- Is there a Schubert-polynomial basis? Numerics are OK!

THANK YOU!

» References

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