Poincaré polynomials and invariant theory

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Hyperplane Arrangements

A **hyperplane arrangement** $\mathcal{A}$ is a union of codimension-1 subspaces of $\mathbb{K}^n$, each one called a **hyperplane**. We write a defining polynomial $Q(\mathcal{A})$ for $\mathcal{A}$ as a product of linear polynomials in n variables, and the arrangement corresponds to the points in $\mathbb{K}^n$ for which $Q(\mathcal{A})$ vanishes.

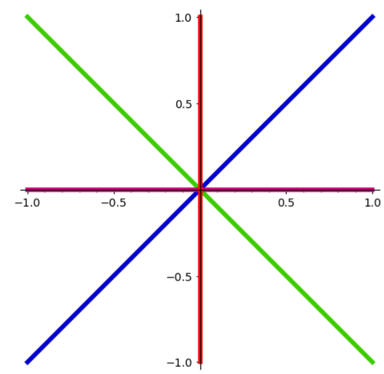

Figure 1: A picture of $\mathcal{A}$ with $Q(\mathcal{A}) = x_1x_2(x_1+x_2)(x_1-x_2)$

Example 1. *Let $Q(\mathcal{A}) = x_1x_2(x_1+x_2)(x_1-x_2)$. There are 4 linear factors, so there are 4 hyperplanes.*

We want to study arrangements, so we associate combinatorial and algebraic objects to an arrangement $\mathcal{A}$.

Definition 1. *The* ***lattice of flats*** *$L(\mathcal{A})$ of $\mathcal{A}$ is the partially ordered set*

$$P = (S, \leq) = (\textit{subspaces of } \mathcal{A}, \supseteq).$$

Define the ***Möbius function***

$$\mu(\mathbb{R}^2, F) = \begin{cases} 1 & F = \mathbb{R}^2 \\ -\sum_{G<F} \mu(\mathbb{R}^2, G) & \textit{else} \end{cases}$$

To each lattice of flats, we associate the ***Poincaré polynomial***[1]

$$\pi(L(\mathcal{A}); t) = \sum_{F \in L(\mathcal{A})} \mu(\mathbb{R}^2, F)(-t)^{\operatorname{codim} F}.$$

[1] If you know the *characteristic polynomial, this is similar, but different.*

Example 2. *The Hasse diagram of $L(\mathcal{A})$ is shown at right, for $\mathcal{A}$ as in Example 1. We have $\pi(L(\mathcal{A}); t) = 1 + 4t + 3t$. To compute $L(\mathcal{A})$ it is sometimes helpful to think of the normal vectors to the subspace. For example,* $\ker([1,0]) = \operatorname{span}[0,y]^T$ *and* $\ker([1,-1]) = \operatorname{span}[x,x]^T$*, and*

$$\ker \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$$

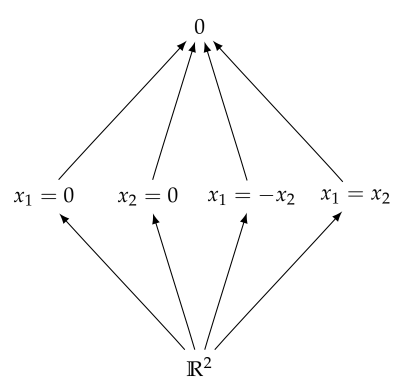

Figure 2: The lattice of flats of the arrangement pictured above.

Polya Theory

Definition 2. *The* ***powersum symmetric function*** p_k *is defined to be*

$$p_k = x_1^k + x_2^k + \cdots \in \lim_{\rightarrow} \mathbb{Q}[x_1, \ldots, x_n].$$

It is a power series in infinitely many variables of bounded degree. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$ *with all parts nonzero. Then define*

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} p_{\lambda_3} \cdots p_{\lambda_\ell}.$$

We also define the ***monomial symmetric function*** m_λ *as*

$$m_\lambda = \sum_{\operatorname{sort}(\alpha)=\lambda} x^\alpha$$

where α *is a sequence of integers and* $\operatorname{sort}(\alpha)$ *is the arrangement of* α *into a weakly decreasing sequence.*

Let $G \leq S_n$. To $g \in G$, we can associate a partition $\rho(g)$ by considering the length of the cycles in g. The partition $\rho(g)$ is constant on S_n conjugacy classes $[g]$.

Definition 3. *The* ***cycle index symmetric function*** *is*

$$Z_G = \frac{1}{|G|} \sum_{g \in G} p_{\rho(g)}.$$

Example 3. *Consider the subgroup* $G = \mathbb{Z}/2 \times \mathbb{Z}/2 \leq S_4$*. The possible* $\rho(g)$ *are* 1111 *and* 211 *and* 22*. The cycle index symmetric function of* G *is*

$$Z_G = \frac{1}{4}\left(p_1^4 + 2p_2p_1^2 + p2^2\right).$$

Let X^S denote the set of all functions $S \rightarrow \{x_1, x_2, x_3, \ldots\}$. Let X^S/G denote the set of orbits of G acting on the variables x_i. We are supposed to think of a single function in X^S as a “coloring" of the set S. Then an orbit in X^S/G is a coloring “up to symmetry." We associate the monomial $x^{\mathcal{O}}$ to each orbit.

Definition 4. *The* ***pattern inventory symmetric function*** *is*

$$F_G = \sum_{\mathcal{O} \in X^S/G} x^{\mathcal{O}}.$$

The definition of Z_G is useful because of the following theorem.

Theorem 1 (Polya)**.** $Z_G = F_G$.

In other words, the expansion into monomials of Z_G enumerates colorings of a set with G symmetry.

Example 4. *Suppose we have* 4 *flavors of donuts. How many ways are there to distribute* 2 *donuts to Alice and* 2 *donuts to Bob? Let*

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2.$$

This is the symmetry group of giving donuts to Alice and Bob because they don't care in which order they have their donuts, but it does matter who gets which. We expand

$$Z_G = \frac{1}{4}\left(p_1^4 + 2p_2p_1^2 + p_2^2\right) = 6m_{1111} + 4m_{211} + 3m_{22} + 2m_{31} + m_4.$$

Thus, there are 6 *ways* ($= \binom{4}{2}$) *to give Alice and Bob* 4 *different flavored donuts, and* 5 *ways to give them two different donuts. For suppose we have glazed (G) and chocolate frosted (F). Then the possibilities are listed at right.*

Alice	Bob
FF	GG
FG	FG
GG	FF.

Figure 3: Possible ways to distribute 2 glazed and 2 frosted donuts, counted by $3m_{22}$

Alice	Bob
FG	GG
GG	FG.

Figure 4: Possible ways to distribute 3 glazed and 1 frosted donuts, counted by $2m_{31}$.

Invariant Theory

Let a group G act on a ring R. The fundamental object[2] of study in (classical) invariant theory is the **invariant ring**

$$R^G = \{r \in R : g \cdot r = r\}.$$

[2] For tools and techniques used to study this ring, see Derksen-Kemper's *Computation Invariant Theory*.

Example 5. $\mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n} = \{$*symmetric polynomials*$\}$

Example 6. $\mathbb{C}[x_1, x_2, \ldots]^{S_n} = \mathbb{C}[p_1, p_2, p_3, \cdots]$, *a polynomial ring in infinitely many variables.*

The **Reynolds operator** $\mathfrak{R}$ is a projection $R \to R^G$.

$$\mathfrak{R}(x) = \frac{1}{|G|}\sum_{g \in G} g \cdot x.$$

Example 7. *Consider* $\mathfrak{R} : \mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}$. *Then*

$$\mathfrak{R}(1) = \frac{1}{n!}\sum_{g \in S_n} g \cdot 1 = \frac{n!}{n!} = 1.$$

and

$$\mathfrak{R}(x_1) = \frac{1}{n!}\sum_{g \in S_n} g \cdot x_1 = \frac{x_1 + x_2 + x_3}{3}$$

and

$$\mathfrak{R}(x_1 + x_2 + \cdots + x_n) = \frac{1}{n!}\sum_{g \in S_n} g \cdot (x_1 + x_2 + \cdots + x_n) = x_1 + x_2 + \cdots + x_n$$

Plethysm

Plethysm is an algebraic operation which can be thought of as "substituting the monomials of a function in for the variables in a symmetric function another."[3]

[3] The definition I'll give here is as written in Macdonald, but you should also see Loehr-Remmel's *A Computational and Combinatorial Exposé of Plethystic Calculus* to learn more and actually use it.

Definition 5. *Suppose $g = \sum_\alpha u_\alpha x^\alpha$. Define y_i as*

$$\prod_i (1 + y_i t) = \prod_i (1 + x^\alpha t)^{u_\alpha}.$$

Then define the ***plethysm***

$$f[g] = f(y_1, y_2, y_3, \ldots).$$

Example 8. *Let $g = 1 + q$. Let*

$$\begin{aligned} f &= p_2 p_1 \\ &= (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) \\ &= x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + \cdots \end{aligned}$$

Then

$$(1 + y_1 t)(1 + y_2 t) \cdots = (1 + t)(1 + qt)$$

so $y_i = 0$ for $i > 2$ and

$$1 + (y_1 + y_2)t + y_1 y_2 t^2 = 1 + (1 + q)t + qt^2.$$

So $y_1 = 1$ and $y_2 = q$ (or the other way around, but f a symmetric function so

$$\begin{aligned} f(y_1, y_2, 0, \ldots) &= f(y_2, y_1, 0, \ldots) \\ &= y_1^3 + y_2^3 + y_1^2 y_2 + y_2^2 y_1 \\ &= 1 + q^3 + q + q^2 \end{aligned}$$

Alternatively we could use the following.

Definition 6. *The operation of* ***plethysm*** *is uniquely defined by*

1. $p_k[p_m] = p_{km}$,
2. $p_k[f \pm g] = p_k[f] \pm p_k[g]$, *(note that this is particular to p_k)*
3. $p_k[f \cdot g] = p_k[f] \cdot p_k[g]$,
4. $(f \pm g)[h] = f[h] \pm g[h]$, *and*
5. $(f \cdot g)[h] = f[h] \cdot g[h]$

Example 9. *Let $f = \frac{1}{4}\left(p_1^4 + 2p_2 p_1^2 + p_2^2\right) = 6m_{1111} + 4m_{211} + 3m_{22} + 2m_{31} + m_4$. Then*

$$\begin{aligned} f[1+q] &= \frac{1}{4}\left((1+q)^4 + 2(1+q^2)(1+q)^2 + (1+q^2)^2\right) \\ &= 3q^2 + 2(q^3 + q) + (1 + q^4). \end{aligned}$$

Conjecture 1 (Foulkes). *The quantity $h_a[h_b] - h_b[h_a]$ is Schur positive.*

Open Problem 1. *Combinatorially interpret*[4] *the coefficients in the Schur expansion of $s_\lambda[s_\mu]$.*

[4] Some cases are known, for example $s_k[s_2]$ is the sum over the Schur functions of all partitions of $2k$ into even parts. For a more detailed list of known cases, see *The Mystery of Plethysm Coefficients* by Colmenarejo-Orellana-Saliola-Schilling-Zabrocki.