Equivariant Kazhdan–Lusztig theory of paving matroids

by Trevor K. Karn (U. Minnesota)
(joint with George Nasr, Nick Proudfoot, and Lorenzo Vecchi)
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A classical story

Our story

The nitty-gritty

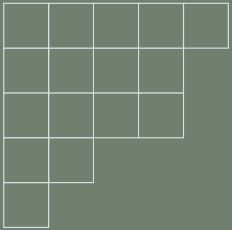
Proof ideas

A classical story

A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n.

Example

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A Young tableau T is a filling of a Ferrers diagram by positive integers. T is standard if it is filled by $\{1, 2, \ldots, n\}$ and increasing in rows and columns. Define f^λ as the number of standard tableaux of shape λ.

Example

One of $f^{(5,4,4,2,1)} = 549120$ standard Young tableaux:

1	6	10	13	16
2	7	11	14	
3	8	12	15	
4	9			
5				

Fact

Fix n. Then

$$\sum_{\lambda \vdash n} (f^{\lambda})^2 = n!$$

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Proof 1:

The Robinson-Schensted bijection:

pairs of standard tableaux of same shape $\longleftrightarrow$ symmetric group $\mathfrak{S}_n$

Fact

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$$\dim S^\lambda = f^\lambda.$$

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Fact

Let $d_1, d_2, \ldots, d_r$ be the dimensions of the irreducible complex representations of a finite group. Then

$$\sum_i d_i^2 = |G|.$$

Fact

Fix n. Then

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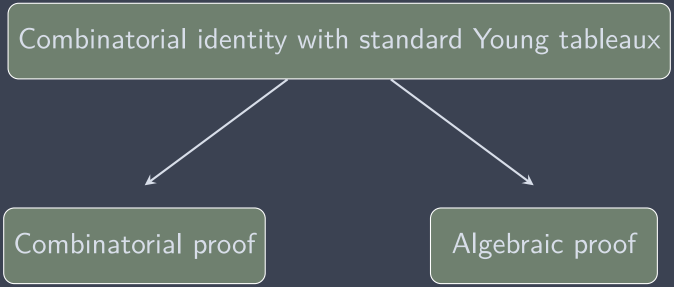
Proof 2:

$$\sum_{\lambda} (f^\lambda)^{\,2} = \sum_i d_i^2 = |G| = |\mathfrak{S}_n| = n!$$

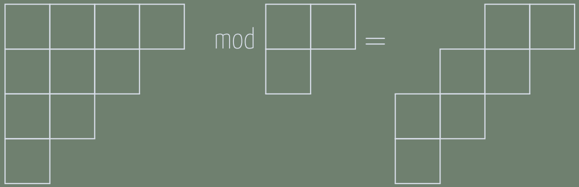

Combinatorial identity with standard Young tableaux
Combinatorial proof
Algebraic proof

Our story

A <u>skew partition</u> λ/μ is a pair of partitions where the diagram of μ is contained in the diagram of λ

Example

If $\lambda = (4,3,2,1)$ and $\mu = (2,1)$ then λ/μ has diagram



A skew tableau T is a filling of a skew diagram by positive integers. T is standard if it is filled by $\{1, 2, \ldots, |\lambda| - |\mu|\}$ and increasing in rows and columns. Define $f^{\lambda/\mu}$ as the number of standard skew tableaux of shape λ/μ.

Example

Two of $f^{(4,3,2,1)/(2,1)} = 272$ standard skew tableaux:

		1	2
	3	4	
5	6		
7			

		3	7
	2	6	
1	5		
4			

Theorem (Lee, Nasr, Radcliffe '21)

Let $P_M(t)$ be the matroid Kazhdan–Lusztig polynomial of M, a rank-k, sparse paving matroid with groundset $[n]$ and circuit hyperplanes $\mathcal{CH}$. The t^i coefficient in $P_M(t)$ is

$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

where

$$\lambda = \left[n-2i, (k-2i+1)^i\right], \mu = \left[(k-2i-1)^i\right]$$

$$\lambda' = \left[(k-2i+1)^{i+1}\right], \mu' = \left[k-2i, (k-2i-1)^{i-1}\right]$$

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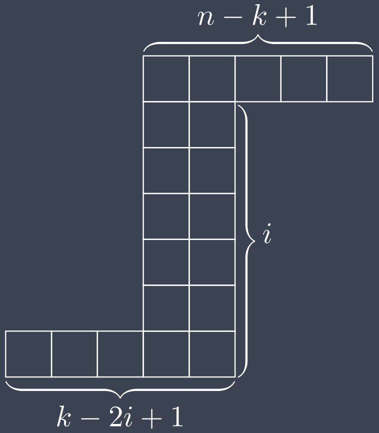
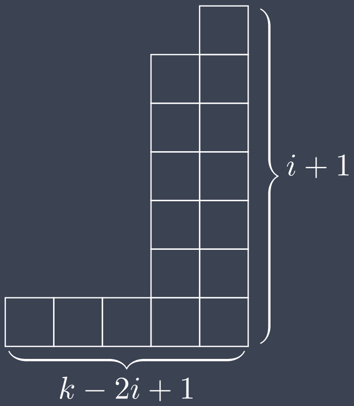

$n-k+1$
i
$k-2i+1$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$k-2i+1$

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Proof 1 (LNR '21): Combinatorial argument with recursion.

Definition

The skew Specht module $S^{\lambda/\mu}$ is

$$S^{\lambda/\mu} = \bigoplus_{\nu} (S^{\nu})^{\oplus c^{\lambda}_{\mu,\nu}}$$

where $c^{\lambda}_{\mu,\nu}$ are Littlewood–Richardson coefficients.

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$S^{\lambda/\mu}$ are (reducible) $\mathfrak{S}_n$ representations and

$$\dim S^{\lambda/\mu} = f^{\lambda/\mu}.$$

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Proof 1 (LNR '21): Combinatorial argument with recursion.
Proof 2 (KNPV '23): $\dim$(skew Specht module from M).

» Example: $U_{3,12}$

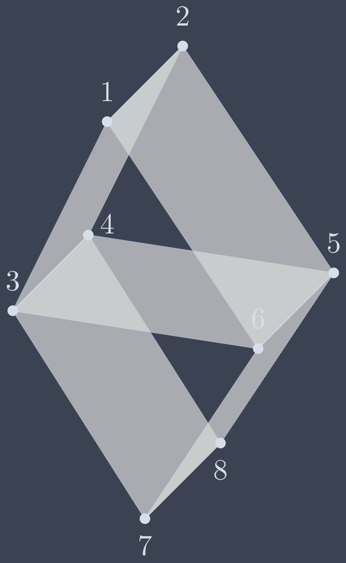
$\lambda/\mu =$

$$\mathcal{CH} = \emptyset$$

$$f^{(10,2)} = 54$$

$$\boxed{P_{U_{3,12}}(t) = 1 + 54t}$$

» Example: Vámos matroid



$\lambda = [6,3],\ \mu = [1] \longrightarrow$

$\lambda' = [3,3]\ , \mu' = [2] \longrightarrow$

$$|\mathcal{CH}| = 5$$

$$f^{\lambda/\mu} - 5f^{\lambda'/\mu'} = 48 - 15 = 33$$

$$\boxed{P_V(t) = 1 + 33t}$$

» Example: Projective plane over $\mathbb{F}_3$

$\lambda/\mu =$ 

$$\mathcal{CH} = \emptyset$$

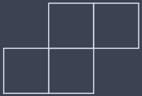
$$f^{\lambda/\mu} \quad = 65 \quad \neq 0$$

$$\boxed{P_M(t) = 1}$$

» Example: Projective plane over $\mathbb{F}_3$

$\lambda/\mu =$ 

$$|\mathcal{SH}| = 13$$

$\lambda'/\mu' =$ 

$$f^{\lambda/\mu} - 13 f^{\lambda'/\mu'} = 65 - 13 * 5 = 0$$

$$\boxed{P_M(t) = 1}$$

Theorem

Let $P_M(t)$ be the matroid Kazhdan–Lusztig polynomial of M, a rank-k, (arbitrary!) paving matroid with groundset $[n]$ and nontrivial stressed hyperplanes $\mathcal{SH}$. The t^i coefficient in $P_M(t)$ is

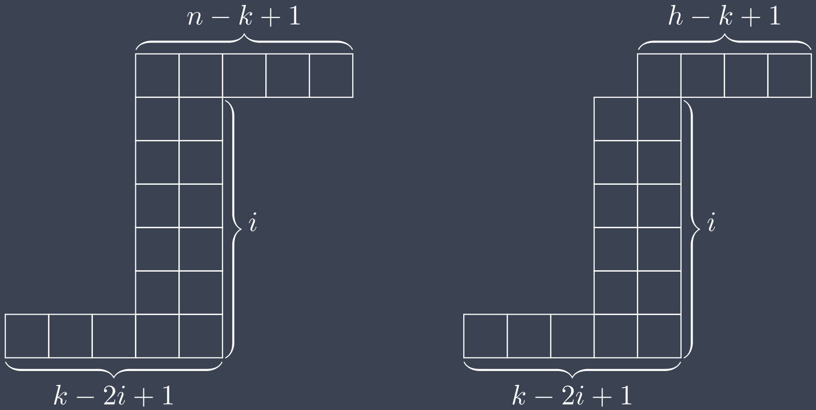
$$f^{\lambda/\mu} - \sum_{H \in \mathcal{SH}} f^{\lambda'(|H|)/\mu'}$$

where

$$\lambda = \left[n-2i, (k-2i+1)^i\right], \mu = \left[(k-2i-1)^i\right]$$

$$\lambda'(h) = \left[h-2i+1, (k-2i+1)^i\right], \mu' = \left[h-2i, (k-2i-1)^{i-1}\right]$$

Proof: Our proof of LNR's theorem implies this more general result


$n-k+1$
i
$k-2i+1$
$h-k+1$
i
$k-2i+1$

Certain $P_M(t)$ in terms of standard skew Young tableaux

→ Combinatorial proof [LNR21]

→ Algebraic proof [KNPV23]

The nitty-gritty

» What do I owe you?

- Matroids
  - Circuits and stressed hyperplanes
  - (Sparse) paving
- Kazhdan–Lusztig polynomials
- How $S^{\lambda/\mu}$ arises

"Definition" 1

A matroid $M = (E, \mathcal{B})$ is a finite set E (called the **groundset**) together with $\mathcal{B} \subseteq 2^E$ satisfying some axioms combinatorially modeling choices of bases for a vector space.

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Alternatively...

"Definition" 2

A matroid $M = (E, \mathcal{C})$ is a ground set E together with $\mathcal{C} \subseteq 2^E$ satisfying some axioms modeling minimal linear dependence of vectors.

Bases $\longleftrightarrow$ maximal independent sets

Circuits $\longleftrightarrow$ minimal dependent sets

Example

The **uniform matroid** $U_{k,n}$ models n-many k-dimensional vectors in general position

Bases $\longleftrightarrow$ any set of k-many vectors

Circuits $\longleftrightarrow$ any set of $k+1$-many vectors

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Bases $\longleftrightarrow$ any set of k-many vectors

Circuits $\longleftrightarrow$ any set of $k+1$-many vectors

Example of the example

$U_{3,12}$ corresponds to 12 generic vectors in $\mathbb{R}^3$. One choice of basis is $\{e_1, e_2, e_3\}$. On the other hand $\{e_1, e_2, e_3, v\}$ is dependent for any $v \in \mathbb{R}^3$.

The combinatorial model for: vectors $\rightarrow$ groundset elements
subspaces $\rightarrow$ **flats**

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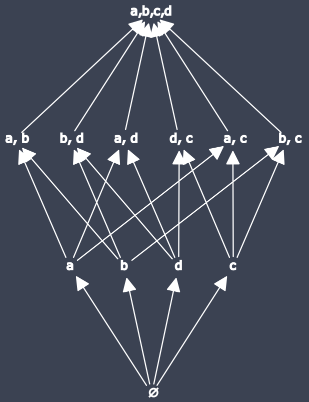


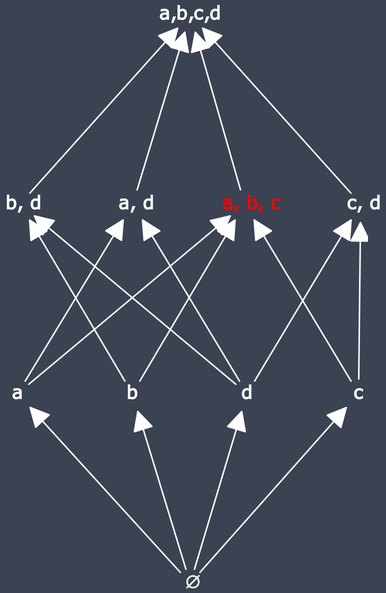
Flats form a ranked lattice L. Define $r(M) = r(L) = k$.
Rank-$(k-1)$ flats are **hyperplanes.** A **circuit hyperplane** is both.

M is a paving matroid if all circuits are at least size $k = r(M)$

A paving matroid is sparse if the set $\mathcal{CH}$ of circuit hyperplanes satisfies $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

A circuit hyperplane is the prototypical example of...

a stressed hyperplane H of a rank-k matroid has every k-subset a circuit.


a,b,c,d
b, d
a, d
a, b, c
c, d
a
b
d
c
∅

Conjecture (Mayhew, Newman, Welsh, Whittle '11)

Asymptotically almost all matroids are sparse paving

($\Rightarrow$ paving)

Theorem (Pendavingh, van der Pol '15)

Asymptotically logarithmically almost all matroids are sparse paving

» What do I owe you?

Matroids ✓

- Circuits and stressed hyperplanes ✓
- (Sparse) paving ✓

Kazhdan–Lusztig polynomials

How $S^{\lambda/\mu}$ arises

In order to define P_M, first define

$$\chi_M(t) = \sum_{F \in L(M)} \mu(\overline{\emptyset}, F) t^{k-r(F)}$$

where μ is the Möbius function.

Definition/Theorem (Elias, Proudfoot, Wakefield '16)

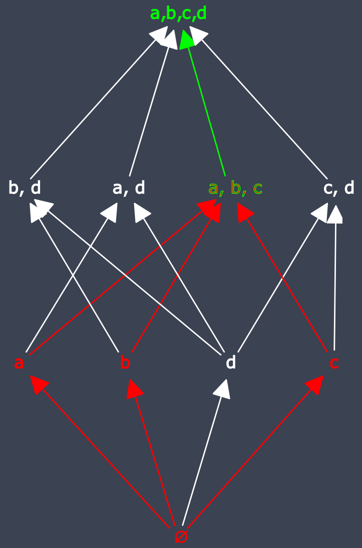
Fix M. There exists a unique polynomial $P_M(t)$ satisfying:

$$P_M(t) = 1 \text{ if } r(M) = 0,$$

$$\deg P_M(t) < r(M)/2 \text{ when } r(M) > 0,$$

$$t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} P_{M_F}(t) \chi_{M^F}(t).$$

M_F and M^F



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- Matroids ✓
  - Circuits and stressed hyperplanes ✓
  - (Sparse) paving ✓
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e.g. $gB \in \mathcal{B}$ for all $g \in W$ and $B \in \mathcal{B}$

gF is another flat of the same rank

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Definition/Theorem (Gedeon, Proudfoot, Young '17)

Let $W \curvearrowright M$ be an equivariant matroid. Then there exists a unique $P_M^W(t)$ with

If $r(M) = 0$, then $P_M^W(t)$ is $\mathbb{1}_W\, t^0$

If $r(M) > 0$, then $\deg P_M^W(t) < r(M)/2$

$$t^{r(M)} \overline{P}_M^W(t) = \sum_{[F] \in L(M)/W} \operatorname{Ind}_{W_F}^W \left(P_{M_F}^{W_F}(t) \otimes \chi_{M^F}^{W_F} \right)$$

$\varphi : W' \to W$ a homom. then $P_M^{W'}(t) = \varphi^* P_M^W(t)$

where W_F denotes the stabilizer of F.

Compare:

$$t^{r(M)}\overline{P_M}(t) = \sum_{F \in L(M)} P_{M_F}(t)\chi_{M^F}(t)$$

and

$$t^{r(M)}\overline{P}_M^W(t) = \sum_{[F] \in L(M)/W} \mathrm{Ind}_{W_F}^W \left(P_{M_F}^{W_F}(t) \otimes \chi_{M^F}^{W_F} \right).$$

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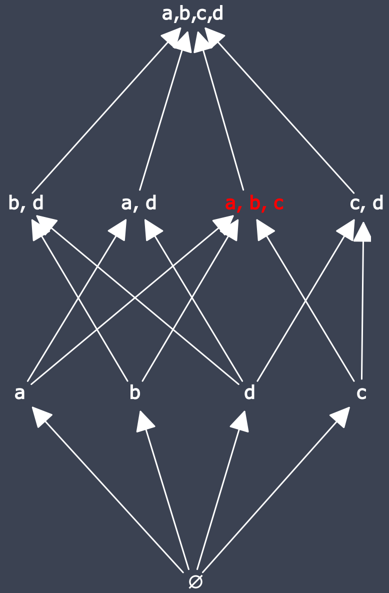
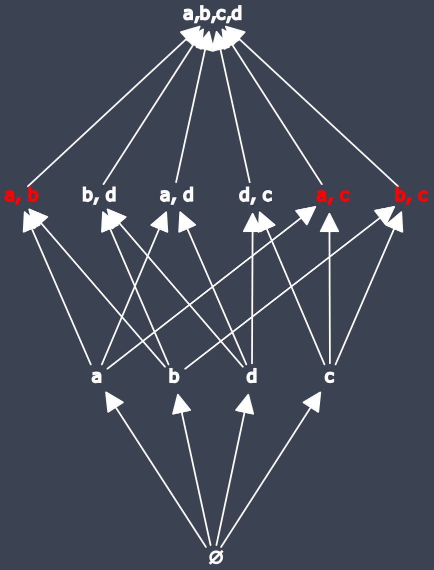
Proof ideas

Theorem (Ferroni, Nasr, Vecchi '21)

Let $M = (E, \mathcal{B})$ be a matroid with stressed hyperplane H. The operation of relaxation at H forms a new matroid $\tilde{M} = (E, \tilde{\mathcal{B}})$ with bases

$$\tilde{\mathcal{B}} = \mathcal{B} \sqcup \{S \subseteq H : |S| = k\}.$$


a,b,c,d
b, d
a, d
a, b, c
c, d
a
b
d
c
∅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Theorem (Ferroni, Nasr, Vecchi '21)

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Fact

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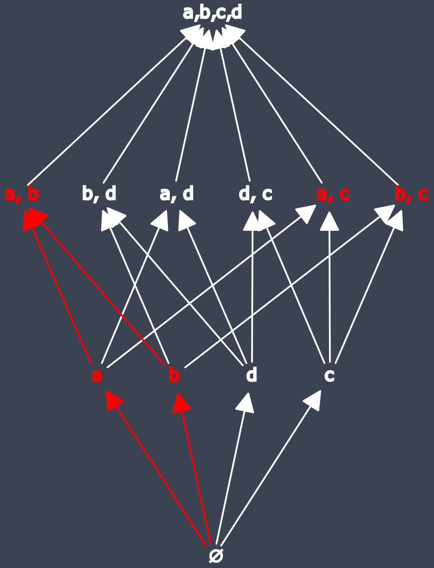
Theorem (Ferroni, Nasr, Vecchi '21)

If M is a paving matroid with $|E| = n$ and has exactly λ_h-many stressed hyperplanes of size h, then

$$P_M(t) = P_{U_{k,n}}(t) - \sum_{h \geq k} \lambda_h \cdot p_{k,h}.$$

» Idea of the proof



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Theorem (K.-Nasr-Proudfoot-Vecchi '23)

There exists an equivariant polynomial $p_{k,h}^{\mathfrak{S}_h}$ such that

$$P_M^W(t) = P_{\widetilde{M}}^W(t) - \operatorname{Ind}_{W_H}^W \operatorname{Res}_{W_H}^{\mathfrak{S}_h} p_{k,h}^{\mathfrak{S}_h}$$

Theorem (K.-Nasr-Proudfoot-Vecchi '23)

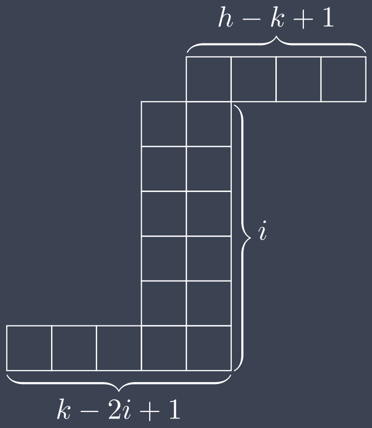
The coefficients of t^i are

$$\{t^i\}p_{k,h}^{\mathfrak{S}_h} = S^{\lambda'/\mu'}$$

where $\lambda', \mu' \vdash h$ are:

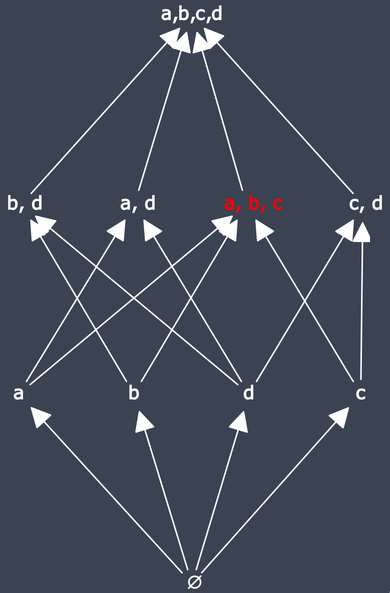
$\lambda' = h - 2i + 1, (k - 2i + 1)^i$ and

$\mu' = k - 2i, (k - 2i - 1)^{i-1}$


$h - k + 1$
i
$k - 2i + 1$

» Idea of proof

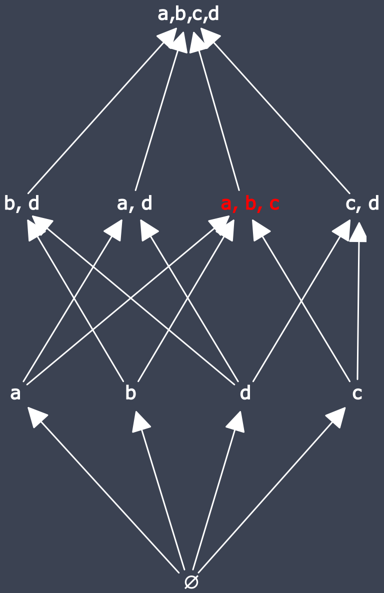
Relax $U_{k-1,h}^{\mathfrak{S}_h} \oplus U_{1,1}$ to $U_{k,h+1}^{\mathfrak{S}_{h+1}}$.



» Idea of proof

Relax $U^{\mathfrak{S}_h}_{k-1,h} \oplus U_{1,1}$ to $U^{\mathfrak{S}_{h+1}}_{k,h+1}$.

$$P_{M_1 \oplus M_2}(t) = P_{M_1}(t) P_{M_2}(t)$$

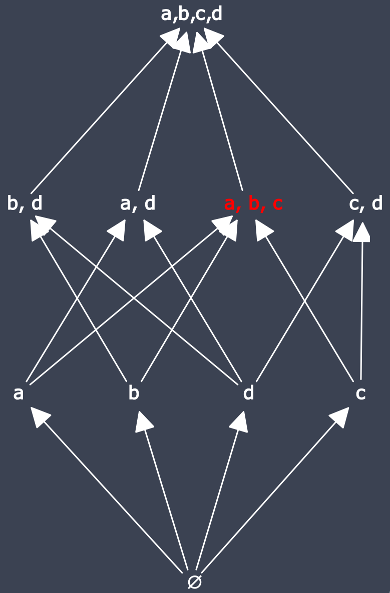


» Idea of proof

Relax $U^{\mathfrak{S}_h}_{k-1,h} \oplus U_{1,1}$ to $U^{\mathfrak{S}_{h+1}}_{k,h+1}$.

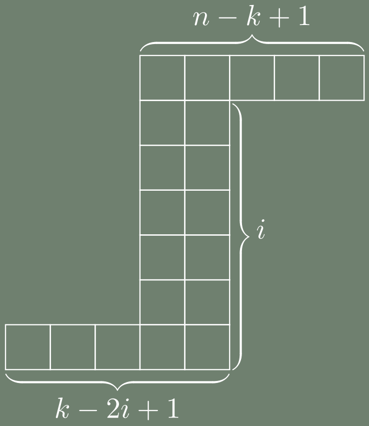
$P_{M_1 \oplus M_2}(t) = P_{M_1}(t) P_{M_2}(t)$

$p^{\mathfrak{S}_h}_{k,h}$ depends only on k, h, so one example is enough.



Theorem (Gao, Xie, Yang '21)

Every coefficient of t^i in $P^{\mathfrak{S}_n}_{U_{k,n}}(t)$ is given by the skew Specht module of shape



Combine:

M is paving $\Leftrightarrow$ a sequence of relaxations makes it $U_{k,n}$

Theorems (K.-Nasr-Proudfoot-Vecchi '23)

$$P_M^W(t) = P_{\widetilde{M}}^W(t) - \mathrm{Ind}_{W_H}^W \mathrm{Res}_{W_H}^{\mathfrak{S}_h} p_{k,h}^{\mathfrak{S}_h}$$

and coefficients of $p_{k,h}^{\mathfrak{S}_h}$ are $S^{\lambda(h)/\mu}$

Theorem (Gao, Xie, Yang '21)

Coefficients of $P_{U_{k,n}}^{\mathfrak{S}_n}(t)$ are $S^{\lambda/\mu}$

$$\dim(S^{\lambda/\mu}) = f^{\lambda/\mu}$$

to obtain...

Theorem

Let $P_M(t)$ be the matroid Kazhdan–Lusztig polynomial of M, a rank-k, arbitrary paving matroid with groundset $[n]$ and nontrivial stressed hyperplanes $\mathcal{SH}$. The t^i coefficient in $P_M(t)$ is

$$f^{\lambda/\mu} - \sum_{H \in \mathcal{SH}} f^{\lambda'(|H|)/\mu'}$$

where λ/μ, λ'/μ' are as before.

THANK YOU!

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