Symmetry, Stability, and interactions with Computation

Exploring permutation representations

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1 Overview

Today, I want to describe a computational tool I have been using to collect data on conjectures. If I was a physicist I might describe this as a piece of lab equipment.

My goal today is twofold. First, I want to describe this approach, in case you all find it helpful in your own work. Secondly, I would love to hear if anyone has thoughts on where this tool might be useful or how to improve it.

1. The problem I want to solve
2. A linear programming approach
3. Extensions?

Assumptions: For the rest of the talk, V will be a finite dimensional or graded vector space over $\mathbb{C}$. For any set X, $\mathbb{C}[X]$ will denote the vector space with basis indexed by X.

2 The problem I want to solve.

A representation $\rho : \mathfrak{S}_n \to \mathrm{GL}(V)$ is called a permutation representation if $\rho(G) \subseteq \mathfrak{S}_n \leq \mathrm{GL}(V)$. Equivalently, this means we could write every $\rho(g)$ as a matrix with a single nonzero entry in every row/column, with the nonzero entry being a 1. I think of this as a representation given by the action of a group on a set, and extending that action to be on the indices of a vector space. (This is Josue's faithful action set up from yesterday.) One thing that will come up later is that the character value of any conjugacy class is exactly the number of fixed points. So in a permutation representation, all the character values are positive.

Example 1. *Consider $\mathfrak{S}_4$ acting on the labels of a square, up to dihedral action.*

$$\begin{array}{ccc} 1 \text{---} 2 & 1 \text{---} 2 & 1 \text{---} 3 \\ | \quad\;\; | & | \quad\;\; | & | \quad\;\; | \\ 3 \text{---} 4 & 4 \text{---} 3 & 4 \text{---} 2 \end{array}$$

Since $[\mathfrak{S}_4 : D_4] = 3$, we see that there are three distinct labelings of the square, determined by the number which is not adjacent to 1. We think of this as acting on the cosets $\mathfrak{S}_4/D_4$. Thus, $\mathbb{C}[\mathfrak{S}_4/D_4]$ carries an $\mathfrak{S}_4$-representation.

Notice that

$$\mathbb{C}[\mathfrak{S}_4/D_4] \cong \mathbb{1}_{D_4} \otimes_{\mathbb{C}[D_4]} \mathbb{C}[\mathfrak{S}_4] \cong \mathbb{1} \uparrow_{D_4}^{\mathfrak{S}_4} .$$

Fact 1. *Every permutation representation looks like* $\mathbb{1}\uparrow_H^G$ *for some* H.

Fact 2. *The Frobenius character map*

$$\mathrm{ch} : \bigcup_n \mathfrak{S}_n \textit{ representations} \to \bigcup_n \Lambda_n$$

is an injective function. It is linear with respect to $\oplus$ *of representations. There are two important linear bases for us. They are both indexed by partitions:*

1. *the complete homogenous basis* h_λ
2. *the power sum basis* p_λ.

Upshot: Any representation (up to isomorphism) is uniquely mapped corresponds to a symmetric function, so if we understand symmetric functions, we understand $\mathfrak{S}_n$-representations (and more!).

To compute $\mathbb{1}\uparrow_H^G$, lets start with an easy case.

Example 2. *It is easy to compute* $\mathrm{ch}\,\mathbb{1}\uparrow_{\mathfrak{S}_n\times\mathfrak{S}_m}^{\mathfrak{S}_{n+m}}$ *in the* h_λ*-basis. It is exactly* $h_n h_m$. *Thus, by the transitivity of induction, we can easily compute* $\mathrm{ch}\,\mathbb{1}\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} = h_\lambda$, *where* $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1}\times\mathfrak{S}_{\lambda_2}\times\cdots\leq\mathfrak{S}_{|\lambda|}$. *Thus, every* h_λ *appears as the Frobenius character of a permutation representation:*

$$h_\lambda = \mathrm{ch}\,\mathbb{C}[\mathfrak{S}_n/\mathfrak{S}_\lambda].$$

Upshot: If a symmetric function f expressed in the h-basis has non-negative integral coefficients (i.e. if it is "h-positive"), then it is a permutation representation with respect to some basis. (Finding the objects which it permutes is a different and difficult question - see for example recent work studying the Chow ring of the boolean matroid from my colleague Robbie Angarone, mathematical sister Anastasia Nathanson, and advisor Vic Reiner [arXiv:2309.14312].)

Example 3. *Let* $\mathfrak{S}_4$ *act on the three dimensional space indexed by labelings of the corners of a square. Then*

$$\mathrm{ch}\,V = h_{22} - h_{31} + h_4.$$

This means we know the irreducible decomposition of V. *I want to point out that this representation was* ***defined as a permutation representation, but is NOT*** h***-POSITIVE!***

This leads into the main question I want to ask:

Question 1. *Given an* $\mathfrak{S}_n$*-representation* V, *is it a permutation representation? If so, what are the groups* H_i *so that*

$$V = \bigoplus_i \mathbb{C}[\mathfrak{S}_n/H_i]?$$

If we can understand the H_i, maybe we can understand the sort of combinatorial objects we should be looking for to index a basis for V - they must have an orbit representative stabilized by H_i.

We have seen that every $\mathfrak{S}_n$-representation V can be encoded as a symmetric function of degree n. h-positivity is a sufficient, but not necessary, condition for V to be a permutation representation. If $\mathrm{ch}\,V$ is not p-positive, then it cannot be a permutation representation, because the coefficients in the p-expansion comes from the character value after applying an appropriate normalizing constant.

h-positive	a permutation rep, sum of $\mathbb{C}[\mathfrak{S}_n/\mathfrak{S}_\lambda]$.
p-non positive	*not* a perm. rep
p-positive and h-non positive	????

This leaves a gap of what to do for p-positive $\operatorname{ch} V$ that are not h-positive. Now we address what to do in such cases.

We have also seen that every permutation representation can be written $\mathbb{1} \uparrow_H^{\mathfrak{S}_n}$.

3 Linear programming approach

Observe the following inequality:

$$\#\{\lambda \vdash n\} \leq \#\{H \leq \mathfrak{S}_n\}.$$

This means that there must be linear dependencies among the symmetric functions

$$\{\operatorname{ch} \mathbb{1} \uparrow_H^{\mathfrak{S}_n}\} \subseteq \Lambda.$$

(There are at least the $\mathfrak{S}_\lambda$'s together with cyclic groups.

Example 4. $\operatorname{ch} \mathbb{C}[\mathfrak{S}_4/D_4] + \operatorname{ch} \mathbb{C}[\mathfrak{S}_4/\mathfrak{S}_{31}] = \operatorname{ch} \mathbb{C}[\mathfrak{S}_4/\mathfrak{S}_{22}] + \operatorname{ch} \mathbb{C}[\mathfrak{S}_4/\mathfrak{S}_4]$. *Thus, there are two combinatorial bases for this representation:*

$$\{\textit{labelings of squares}\} \cup \{\binom{[4]}{3}\}$$

and

$$\binom{[4]}{2} \cup \{1\}$$

Additional algebraic structure (e.g. a grading) might make one of these more natural than the other.

Remark 1. *The Grothendiek ring of permutation representations with respect to addition $\oplus$ and multiplication $\otimes$ is called the* Burnside ring $\mathfrak{B}$. *The linear relation above is an example of a linear relation in a presentation of $\mathfrak{B}$, but there are also multiplicative relations. It would be a bit cumbersome to write a nontrivial example, but one can find the nontrivial examples by looking at the kernel of the map from the Burnside ring to the ring of symmetric functions.*

Let $\{H_i\}_{i \in I}$ is a complete list of subgroups of $\mathfrak{S}_n$ up to conjugacy. Let χ denote the vector of character values of V.

$$\begin{array}{ll} \text{maximize} & \mathbf{1}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \chi \\ \text{and} & \mathbf{x} \geq \mathbf{0} \end{array}$$

where $\mathbf{1}$ is the all-ones vector, where

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \chi_1 & \chi_2 & \cdots & \chi_c \\ | & | & & | \end{bmatrix}$$

and χ_i is the vector of character values for $\mathbb{C}[\mathfrak{S}_n/H_i]$. then x_i is the multiplicity of $\mathbb{C}[\mathfrak{S}_n/H_i]$ in V.

Where should $\mathbf{x}$ live? Let $c = \#\{H_i\}$. Well maybe we think $\mathbb{Z}^c_{\geq 0}$. But if we expand to $\mathbf{Q}^c$, we might find that we get coefficients with a common denominator suggesting that we should some how find a multiple of our representation. I think Vic will talk about such a case of this happening this afternoon.

I've successfully used this up to S_{10} because iterating over all the subgroups of S_{11} up to conjugugacy takes a while and I haven't needed to go any higher. The linear program is certainly not the problem, it is more about computing χ_i's.

Example 5. *When I plug in V with $\operatorname{ch} V = h_{22} + h_4$ into my implementation of this linear program, I get that the solution $\mathbf{x}$ corresponds to a copy of D_4 and $\mathfrak{S}_{31}$. Looking at the symmetric function tells me that it could also correspond to $\mathfrak{S}_{22}$ and $\mathfrak{S}_4$. So this approach gives us the non-obvious result.*[1]

4 Extensions?

4.1 Other objective functions

The objective function $\mathbf{1}^T\mathbf{x}$ encourages us to maximize the number of H_i we find. If we have reason to believe that one particular is present, we could change our program to be

$$\begin{array}{ll} \text{maximize} & \mathbf{w}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \chi \\ \text{and} & \mathbf{x} \geq \mathbf{0} \end{array}$$

where we increase w_i for the the H_i we expect to occur. For example, if V_0 is a copy of the field, we should either ignore it, or add weight to the trivial representation.

Setting $w_i = 100$ and $w_j = 1$ for $H_i = \mathfrak{S}_4$ and H_j for every other group gives the solution $\mathfrak{S}_{22}$ and $\mathfrak{S}_4$. We knew this from the symmetric function, but this is just a proof of concept.

4.2 Other groups

This approach should work for any group, not just the symmetric group.

4.3 Other symmetric functions

Here we were concerned with h-positivity of symmetric functions. But one could also ask about e-positivity, for example like in the the Stanley-Stembridge conjecture.

[1] Milage may vary - this is only an experimentation tool.