Combinatorics in Kazhdan–Lusztig polynomials of paving matroids

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A classical story

Our story

A classical story

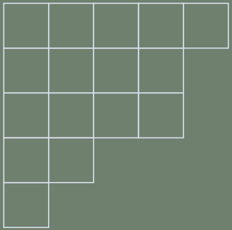
Euclid stated five postulates for rigorous geometry.

We can drop the fifth and still do geometry.

A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n.

Example

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A <u>Young tableau</u> T is a filling of a Ferrers diagram by positive integers. T is <u>standard</u> if it is filled by $\{1, 2, \ldots, n\}$ and increasing in rows and columns. Define f^λ as the number of standard tableaux of shape λ.

Example

One of $f^{(5,4,4,2,1)} = 549120$ standard Young tableaux:

1	6	10	13	16
2	7	11	14	
3	8	12	15	
4	9			
5				

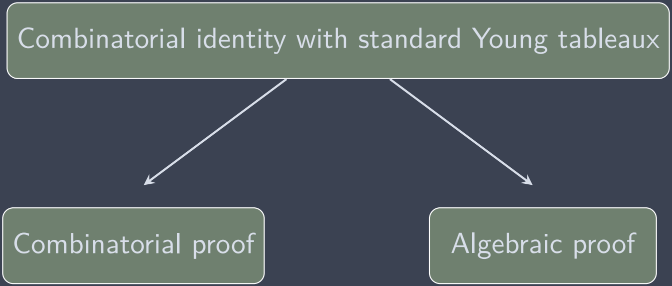
Fact

Fix n. Then
$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Proof 1:
The Robinson-Schensted bijection:

pairs of standard tableaux of same shape $\longleftrightarrow$ symmetric group $\mathfrak{S}_n$

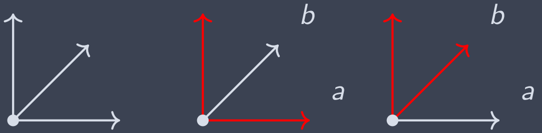
Proof 2:
Uses that irreducible representations of $\mathfrak{S}_n$ are indexed by $\lambda \vdash n$ and have dimension f^λ


Combinatorial identity with standard Young tableaux
Combinatorial proof
Algebraic proof

Our story

Let $v_1, \ldots, v_n$ be vectors in a vector space V. Then any two bases A, B for the span of $v_1, \ldots v_n$ satisfy the following requirements:

1) There must be at least one basis
2) If $a \in A - B$ then there is a $b \in B$ with $(A - a) \cup b$ a basis.



Definition

A **matroid** $M = (E, \mathcal{B})$ is a set E with $\emptyset \neq \mathcal{B} \subseteq 2^E$ such that if $A, B \in \mathcal{B}$ and $a \in A$, there exists $b \in B$ such that

$$(A - a) \cup b \in \mathcal{B}$$

Call $\mathcal{B}$ the bases of the matroid.

Matroids have the combinatorics of vectors without the "zeroth postulate".

» Vocab

Finite set of vectors	Matroids
Maximally independent sets	Bases $\mathcal{B}$
Minimally dependent sets	Circuits $\mathcal{C}$
Dimension of span	Rank
Codimension 1 subspaces	Hyperplanes $\mathcal{H}$

Definition

$\mathcal{CH} = C \cap H$ is the set of circuit hyperplanes.

A hyperplane H is stressed if every subset of H of size $\mathrm{rk}(E)$ is in $\mathcal{C}$. Denote the set of (nontrivial) stressed hyperplanes by $\mathcal{SH}$.

Fact

There is a very large class of matroids called (sparse) paving matroids.

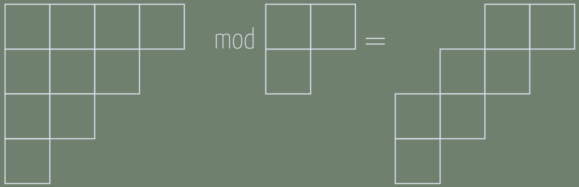
Fact

The matroid Kazhdan–Lusztig polynomial $P_M(t)$ is an interesting polynomial invariant of a matroid M, introduced by Elias, Proudfoot, and Wakefield in 2016.

A <u>skew partition</u> λ/μ is a pair of partitions where the diagram of μ is contained in the diagram of λ

Example

If $\lambda = (4,3,2,1)$ and $\mu = (2,1)$ then λ/μ has diagram



A skew tableau T is a filling of a skew diagram by positive integers. T is standard if it is filled by $\{1, 2, \ldots, |\lambda| - |\mu|\}$ and increasing in rows and columns. Define $f^{\lambda/\mu}$ as the number of standard skew tableaux of shape λ/μ.

Example

Two of $f^{(4,3,2,1)/(2,1)} = 272$ standard skew tableaux:

		1	2
	3	4	
5	6		
7			

		3	7
	2	6	
1	5		
4			

Theorem (Lee, Nasr, Radcliffe '21)

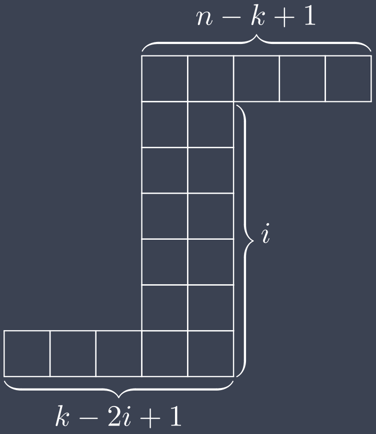
Let M be a rank-k, sparse paving matroid with $E = [n]$ and circuit hyperplanes $\mathcal{CH}$. The t^i coefficient in $P_M(t)$ is

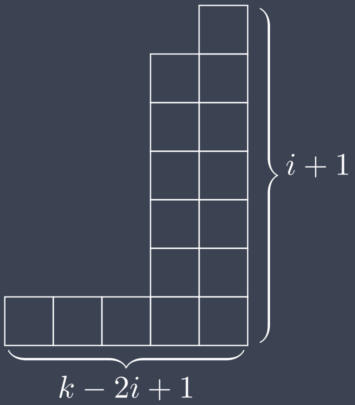
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

where

$$\lambda = [n-2i, (k-2i+1)^i], \mu = [(k-2i-1)^i]$$

$$\lambda' = [(k-2i+1)^{i+1}], \mu' = [k-2i, (k-2i-1)^{i-1}]$$


$n-k+1$
i
$k-2i+1$


$i+1$
$k-2i+1$

Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M, the t^i coefficient in $P_M(t)$ is
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.

Fact

There is a (reducible) $\mathfrak{S}_n$ representation $S^{\lambda/\mu}$ of dimension $f^{\lambda/\mu}$.

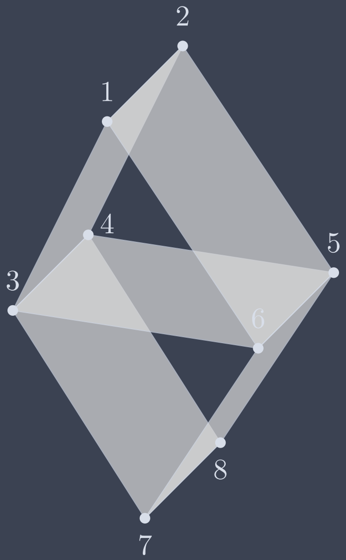
Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M, the t^i coefficient in $P_M(t)$ is

$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.
Proof 2 (KNPV '22): $\dim$(some $S^{\lambda/\mu}$ coming from M).

» Example: Vámos matroid



$$\lambda = [6,3],\ \mu = [1] \longrightarrow$$

$$\lambda' = [3,3]\ , \mu' = [2] \longrightarrow$$

$$|\mathcal{CH}| = 5$$

$$f^{\lambda/\mu} - 5f^{\lambda'/\mu'} = 48 - 15 = 33$$

$$\boxed{P_V(t) = 1 + 33t}$$

» Example: Projective plane over $\mathbb{F}_3$

$$\lambda/\mu = \ydiagram{11,2}$$

$$\mathcal{CH} = \emptyset$$

$$|\mathcal{SH}| = 13$$

$$\lambda'/\mu' = \ydiagram{1+2,2}$$

$$f^{\lambda/\mu} - 13 f^{\lambda'/\mu'} = 65 - 13 * 5 = \;\neq 0$$

$$\boxed{P_M(t) = 1}$$

Theorem

For a (arbitrary!) paving matroid M, the t^i coefficient in $P_M(t)$ is

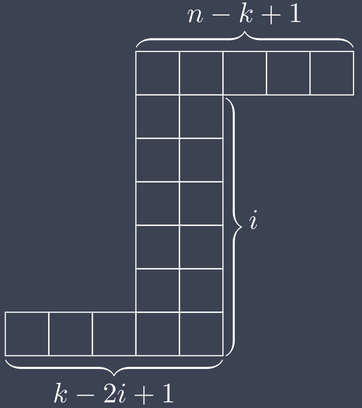
$$f^{\lambda/\mu} - \sum_{H \in \mathcal{SH}} f^{\lambda'(|H|)/\mu'}$$

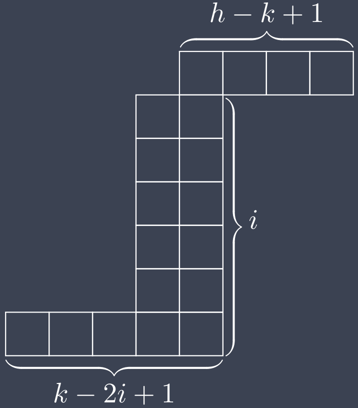
where

$$\lambda = \left[n-2i, (k-2i+1)^i\right], \mu = \left[(k-2i-1)^i\right]$$

$$\lambda'(h) = \left[h-2i+1, (k-2i+1)^i\right], \mu' = \left[h-2i, (k-2i-1)^{i-1}\right]$$

Proof: Our proof of LNR's theorem implies this more general result


$n-k+1$
i
$k-2i+1$


$h-k+1$
i
$k-2i+1$

Certain $P_M(t)$ in terms of standard skew Young tableaux

Combinatorial proof [LNR21]

Algebraic proof [KNPV22]

THANK YOU!

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