Combinatorial formulas for Kazdan-Lusztig polys of paving matroids.
à l'envers

Outline

I) Rep theory background

II) Dimension of rep'ns

III) The Littlewood-Richardson rule

IV) Practice

(I)

Def/ A group representation ρ is a homom. $\rho: G \to GL(V)$

Today: $|G| < \infty$, V f.d. $\mathbb{C}$ vector space

Terminology: call V the rep'n (eventhough def to be homom.)
Idea: G acts on vectors in V $\quad g \cdot v := \rho(g)(v)$

A sub rep'n $W \subseteq V$ is a vector subspace of V such that the action of $g \in G$ on $w \in W$ is still inside W: $g \cdot w \in W \ \forall g, w$

In our setting: V can be decomposed into $\bigoplus W$ where

W are as small as possible, but still subreps. Call these irreps.

Def/ Irrep if can't properly decompose into direct sum of subreps.

N.B.: more subtle if $|G| = \infty$ or V not a $\mathbb{C}$-vector space

If $H \leq G$, can build a G-rep'n from an H rep'n and an H-rep'n from a G-rep'n:

$$H \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} G$$

To me: Res = "forget $G - H$" $\quad$ Ind = $\mathbb{C}[G] \otimes_H V$, but $\exists$ other equivalent descr.

Ex $G = S_3 \curvearrowright \mathbb{C}^4$ by permuting e_1, e_2, e_3, e_4

$$(123)\cdot\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}3\\1\\2\\4\end{bmatrix} \qquad \mathbb{C}^4 = \mathbb{C}^3 \oplus \mathrm{span}_{\mathbb{C}}\{e_4\}$$

$$(13)\cdot\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \in \mathbb{C}^3 \implies \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}\{e_1+e_2+e_3\} \oplus \mathbb{C}\{e_4\}$$

Turns out $\mathbb{C}\left\{\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{b_2}\right\}$ is the other part, and we need both

$$(123)\, b_1 = \begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix} = b_1 - b_2$$

$\mathbb{C}^4\downarrow^{S_3}_{C_3}$ acts the same way $\cong \mathbb{C}^4$

Now $\quad \mathbb{C}[S_3] \otimes_{C_3} \mathbb{C}^4\downarrow^{S_4}_{C_3} \cong \mathbb{C}^8 \quad$ as vector spaces

punchline $\quad V\downarrow^G_H\uparrow^G_H \ncong V$ in general.

Irreducibles of S_n are indexed by partitions $\lambda \vdash n \to S^\lambda$

$\mathbb{C}^4\downarrow^{S_3}_{C_3}\uparrow^{S_3}_{C_3} \longleftrightarrow \quad \square\square\square \times 2, \quad \begin{array}{c}\square\\\square\\\square\end{array} \times 2, \quad \begin{array}{ll}\square&\square\\\square&\end{array} \times 2$

II) Dimensions - the first invariant of a vector space.

Notice from example:
- not all irreps had same dimension
- restricting presered dim
- inducing changes dimension

for $G = S_n$, can use combinatorics to get at dimensions

Def/ A standard Young tableau is a filling of Ferrers diagram that is increasing in rows/columns.

Ex $\quad \begin{array}{ccc}1&2&3\end{array} \qquad \begin{array}{c}1\\2\\3\end{array} \qquad \begin{array}{cc}1&2\\3&\end{array} \qquad \begin{array}{cc}1&3\\2&\end{array}$

<u>Fact</u> $\dim(S^\lambda) = \#\,\mathrm{SYT}$ of shape λ.

<u>Ex</u> $\dim(S^{(3)}) = 1$, $\dim(S^{(1,1,1)}) = 1$, $\dim(S^{(2,1)}) = 2$

$$\dim(\mathbb{C}^4 \downarrow^{S_3}_{C_3} \uparrow^{S_3}_{C_3}) = 2\left(\dim S^{(3)} + \dim S^{(1,1,1)} + \dim S^{(2,1)}\right) = 2\cdot 4 = 8$$

$\Rightarrow \cong \mathbb{C}^8$ as v.s.

How could we predict this?

$$\dim(\mathbb{C}^4 \downarrow^{S_3}_{C_3}) = \dim(\mathbb{C}^4) \quad \text{because we forget some matrices}$$

$$\dim(\mathbb{C}^4 \uparrow^{S_3}_{C_3}) = [S_3 : C_3]\dim(\mathbb{C}^4) = 2\cdot 4 = 8.$$

$$[S_3 : C_3] = \#\text{ cosets of } C_3 \text{ in } S_3 = \frac{|S_3|}{|C_3|} = \frac{6}{3} = 2.$$

more generally, $\dim(V\uparrow_H^G) = [G:H]\dim V$.

Why? $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = V\uparrow_H^G$ $\quad \otimes_{\mathbb{C}[H]}$ means pull elts of $\mathbb{C}[H]$ through. So $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ gen'd by

$$\{g \otimes v : g \in G,\ v \text{ a basis vector of } V\}$$

but if $g \in g'H$, then $g = g'h$ so $g\otimes v = g'h \otimes v = g' \otimes hv$

which can be expanded as a sum

$$\sum_{\substack{g' \text{ a rep.}\\ \text{for each}\\ \text{coset of}\\ H \text{ in } G}} c_{g',v}\, g' \otimes v$$

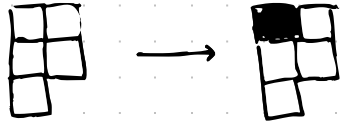
<u>Upshot</u>: Once we know $\dim V$ and $[G:H]$, we know $\dim V\uparrow_H^G$.

III) LR rule

Now define "skew partitions λ/μ". Start with λ and take a μ-shaped bite out of it.

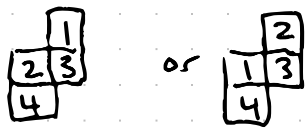
Ex $221/1$

$\lambda = 221, \mu = 1$



Define a std filling of λ/μ as filled by $|\lambda| - |\mu|$ and increasing in rows / columns.

Ex



Define a new rep'n with dimension = # std fillings of λ/μ

$S^{\lambda/\mu}$. <u>not</u> irreducible. In fact

$$S^{\lambda/\mu} = \bigoplus_{\nu} (S^{\nu})^{c^{\lambda}_{\mu,\nu}}$$

where $c^{\lambda}_{\mu,\nu}$ = "Littlewood Richardson coefficients"

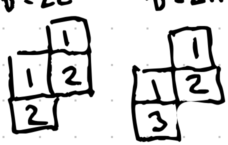
$c^{\lambda}_{\mu,\nu}$ = # LR tableaux of shape λ/μ and content ν.

Def/ LR tableaux of shape λ/μ are a filling of a skew shape which is increasing in cols & weakly increasing in rows, and has "reading word" a ballot sequence.

Read the #'s as if in العربي $\leftarrow$

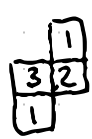
content: a part'n
#1's, #2's, #3's, ...

Ex $\nu = 22$ $\nu = 211$



1
121
1212
always at least as many 1's as 2's

<u>Non ex</u>



$\leftarrow$ bad row

$\uparrow$ bad col

incr. in row/col
<u>not</u> ballot seq.

1
13 $\leftarrow$ no 2
132
1324

Q: What is $S^{3321/22}$?